AMATH/PMATH 331 - Applied Real Analysis

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Contents

Chapter 1

Real Limits, Continuity and Differentiation

1.1 Order Properties in R

THEOREM 1.1.1: Discreteness Property of Z

We state two equivalent definitions.

$$\forall k \in \mathbf{Z}\ \forall n \in \mathbf{Z}\ (k \leq n \iff k < n + 1)$$

$$\forall n \in \mathbf{Z}\ \nexists k \in \mathbf{Z}\ (n < k < n + 1)$$

Proof of: Theorem 1.1.1

Accepted axiomatically, without proof.

DEFINITION 1.1.2: Bounded above, Upper bound

A is **bounded above** (in **R**) when

$$\exists b \in \mathbf{R}\ \forall x \in A\ (x \leq b)$$

We say that b is an **upper bound** for A.

DEFINITION 1.1.3: Bounded below, Lower bound

A is **bounded below** (in **R**) when

$$\exists a \in \mathbf{R}\ \forall x \in A\ (a \leq x)$$

We say that a is a **lower bound** for A.

DEFINITION 1.1.4: Bounded

A is **bounded** when A is both bounded above and below.

DEFINITION 1.1.5: Supremum, Least upper bound, Maximum element

A has a **supremum** (or a **greatest lower bound**) when there exists an element $b \in \mathbf{R}$ such that b is an upper bound for A with $b \leq c$ for every upper bound $c \in \mathbf{R}$ for A. In this case, we say b is the **supremum** (or the **greatest lower bound**) of A and write $b = \sup\{A\}$. When $b = \sup\{A\} \in A$ we also say that b is the **maximum element** of A, and we write $b = \max\{A\}$.

DEFINITION 1.1.6: Infimium, Greatest lower bound, Minimum element

A has an **infimum** (or a **greatest lower bound**) when there exists an element $a \in \mathbf{R}$ such that a is a lower bound for A with $c \leq a$ for every lower bound c for A. In this case, we say a is the **infimum** (or the **greatest lower bound**) of A and write $a = \inf\{A\}$. When $a = \inf\{A\} \in A$ we also say that a is the **minimum element** of A, and we write $a = \min\{A\}$.

EXAMPLE 1.1.7

Let $A = \mathbf{R}_{>0} = (0, \infty) = \{x \in \mathbf{R} \mid x > 0\}$ and $B = [1, \sqrt{2}) = \{x \in \mathbf{R} \mid 1 \leq x < \sqrt{2}\}$.

- A is bounded below, but not above.
- -1 and 0 are both lower bounds for A.
- $\inf\{A\} = 0$
- A has no minimum element, and no maximum element.
- B is bounded both above and below.
- 0 and 1 are both lower bounds for B
- $\sqrt{2}$ and 3 are both upper bounds for B.
- $\inf\{B\} = 1$
- $\sup\{B\} = \sqrt{2}$
- B has a minimum element, namely $\min\{B\} = 1$, but has no maximum element.

THEOREM 1.1.8: The Supremum and Infemum Properties of R

(1) Every non-empty subset of **R** *which is bounded above in* **R** *has a supremum in* **R***.*
(2) Every non-empty subset of **R** *which is bounded below in* **R** *has an infimum in* **R***.*

Proof of: Theorem 1.1.8

Accepted axiomatically, without proof.

THEOREM 1.1.9: Approximation Property of Supremum and Infimum

Let $\emptyset \neq A \in \mathbf{R}$.
(1) $b = \sup\{A\} \implies \forall \varepsilon \in \mathbf{R}_{>0}\ \exists x \in A\ (b - \varepsilon < x \leq b)$
(2) $a = \inf\{A\} \implies \forall \varepsilon \in \mathbf{R}_{>0}\ \exists x \in A\ (a \leq x < a + \varepsilon)$

Proof of: Theorem 1.1.9

We prove (1). Let $b = \sup\{A\}$ and $\varepsilon > 0$. Suppose for a contradiction that there exists no element $x \in A$ with $b - \varepsilon < x$, or equivalently that for all $x \in A$ we have $b - \varepsilon \geq x$. Let $c = b - \varepsilon$. Note that c is an upper bound for A since $x \leq b - \varepsilon = c$ for all $x \in A$. Then, since $b = \sup\{A\}$ and c is an upper bound for A, we have $b \leq c$. However, since $\varepsilon > 0$ we have $b > b - \varepsilon = c$, contradiction. Therefore, there exists $x \in A$ with $b - \varepsilon < x$. Now, choose an element $x \in A$. Then, since $b = \sup\{A\}$, we know that b is an upper bound for A and hence $b \geq x$. Therefore, $b - \varepsilon < x \leq b$, as required.

THEOREM 1.1.10: Well-Ordering Properties of Z in R

(1) Every non-empty subset of $\mathbf{Z}$ *which is bounded above in* $\mathbf{R}$ *has a maximum element.*
(2) Every non-empty subset of $\mathbf{Z}$ *which is bounded below in* $\mathbf{R}$ *has a minimum element.*

Proof of: Theorem 1.1.10

We prove (1). Let A be a non-empty subset of $\mathbf{Z}$ which is bounded above. By Theorem 1.1.8 (1), A has a supremum in $\mathbf{R}$. Let $n = \sup\{A\}$. We must show that $n \in A$. Suppose for a contradiction that $n \notin A$. By Theorem 1.1.9 (using $\varepsilon = 1$), we can choose $a \in A$ with $n - 1 < a \leq n$. Note that $a \neq n$ since $a \in A$ and $n \notin A$, so we have $a < n$. By Theorem 1.1.9 (using $\varepsilon = n - a$) we can choose $b \in A$ with $a < b \leq n$. Since $a < b$ we have $b - a > 0$. Since $n - 1 < a$ and $b \leq n$, we have $1 = n - (n-1) > b - a$. However, we have $(b - a) \in \mathbf{Z}$ with $0 < b - a < 1$, which contradicts Theorem 1.1.1. Therefore, $n \in A$, and hence A has a maximum element.

THEOREM 1.1.11: Floor and Ceiling Properties of Z in R

(1) $\forall x \in \mathbf{R}\ \exists! n \in \mathbf{Z}\ (x - 1 < n \leq x)$.
(2) $\forall x \in \mathbf{R}\ \exists! m \in \mathbf{Z}\ (x \leq m < x + 1)$.

Proof of: Theorem 1.1.11

We prove (1).
Uniqueness. Let $x \in \mathbf{R}$, suppose $n, m \in \mathbf{Z}$ with $x - 1 < n \leq x$ and $x - 1 < m \leq x$. Since $x - 1 < n$ we have $x < n + 1$. Since $m \leq x$ and $x < n + 1$, we have $m < n + 1$, hence $m \leq n$ by Theorem 1.1.1. Similarly, $n \leq m$. Since $n \leq m$ and $m \leq n$, we have $n = m$ as required.
Existence. Let $x \in \mathbf{R}$. First, let us consider the case that $x \geq 0$. Let $A = \{k \in \mathbf{Z} \mid k \leq x\}$. Note that $A \neq \emptyset$ (because $0 \in A$), and A is bounded above by x. By Theorem 1.1.10, A has a maximum element. Let $n = \max\{A\}$. Since $n \in A$, we have $n \in \mathbf{Z}$ and $n \leq x$. Also, note that $x - 1 < n$ since $x - 1 \geq n \implies x \geq n + 1 \implies n + 1 \in A \implies n \neq \max\{A\}$. Thus, for $n = \max\{A\}$, we have $n \in \mathbf{Z}$ with $x - 1 < n \leq x$ as required.
Next, consider the case that $x < 0$. If $x \in \mathbf{Z}$, we can take $n = x$. Suppose that $x \notin \mathbf{Z}$. We have $-x > 0$ so, by the previous paragraph, we can choose $m \in \mathbf{Z}$ with $-x - 1 < -m < x + 1$. Thus, we can take $n = -m - 1$ to get $x - 1 < n < x$.

DEFINITION 1.1.12: Floor, Floor function

Let $x \in \mathbf{R}$. The **floor** of x, denoted by $\lfloor x \rfloor$, is the unique $n \in \mathbf{Z}$ with $x - 1 < n \leq x$. The function $f : \mathbf{R} \to \mathbf{Z}$ given by $f(x) = \lfloor x \rfloor$ is called the **floor function**.

DEFINITION 1.1.13: Ceiling, Ceiling function

Let $x \in \mathbf{R}$. The **ceiling** of x, denoted by $\lceil x \rceil$, is the unique $n \in \mathbf{Z}$ with $x \leq n < x + 1$. The function $f : \mathbf{R} \to \mathbf{Z}$ given by $f(x) = \lceil x \rceil$ is called the **ceiling function**.

THEOREM 1.1.14: Archimedean Properties of Z in R

(1) $\forall x \in \mathbf{R}\ \exists n \in \mathbf{Z}\ (n > x)$.
(2) $\forall x \in \mathbf{R}\ \exists m \in \mathbf{Z}\ (m < x)$.

Proof of: Theorem 1.1.14

Let $x \in \mathbf{R}$. Let $n = \lfloor x \rfloor + 1$ and $m = \lfloor x \rfloor - 1$. Since $x - 1 < \lfloor x \rfloor$, we have $x < \lfloor x \rfloor + 1 = n$ and since $\lfloor x \rfloor \leq x$, we have $m = \lfloor x \rfloor - 1 \leq x - 1 < x$.

THEOREM 1.1.15: Density of Q in R

$$\forall a \in \mathbf{R}\ \forall b \in \mathbf{R}\ \exists q \in \mathbf{Q}\ (a < b \implies a < q < b)$$

Proof of

Let $a, b \in \mathbf{R}$ with $a < b$. By Theorem 1.1.14, we can choose $n \in \mathbf{Z}$ with $n > \frac{1}{b-a} > 0$. Then, $n(b-a) > 1$ and so $nb > na + 1$. Let $k = \lfloor na + 1 \rfloor$. Then we have $na < k \leq na + 1 < nb$ hence $a < \frac{k}{n} < b$. Thus, we can take $q = \frac{k}{n}$ to get $a < q < b$.

1.2 Limit of Sequences in R

DEFINITION 1.2.1: Sequence, Term

For $p \in \mathbf{Z}$, let $Z_{\geq p} = \{k \in \mathbf{Z} \mid k \geq p\}$. A **sequence** in a set A is a function of the form $x : \mathbf{Z}_{\geq p} \to A$ for some $p \in \mathbf{Z}$. Given a sequence $x : \mathbf{Z}_{\geq p} \to A$, the k^{th} **term** of the sequence is the element $x_k = x(k) \in A$, and we denote the sequence x by

$$(x_k)_{k \geq p} = (x_p, x_{p+1}, \ldots)$$

Note that the range of the sequence $(x_k)_{k \geq p}$ is the set $\{x_k\}_{k \geq p} = \{x_k \mid k \geq p\}$.

DEFINITION 1.2.2: Limit, Convergence, Divergence

Let $(x_k)_{k \geq p}$ be a sequence in $\mathbf{R}$. For $a \in \mathbf{R}$ we say that $(x_k)_{k \geq p}$ **converges** to a (or that the **limit** of $(x_k)_{k \geq p}$ is equal to a), and we write $x_k \to a$ (as $k \to \infty$), or we write $\lim_{k \to \infty} x_k = a$, when

$$\forall \varepsilon \in \mathbf{R}_{>0}\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ (k \geq m \implies |x_k - a| < \varepsilon)$$

We say that the sequence $(x_k)_{k \geq p}$ converges (in $\mathbf{R}$) when there exists $a \in \mathbf{R}$ such that $(x_k)_{k \geq p}$ converges to a. We say that $(x_k)_{k \geq p}$ **diverges** (in $\mathbf{R}$) when it does not converge (to any $a \in \mathbf{R}$). We say that $(x_k)_{k \geq p}$ **diverges to infinity**, or that the limit of $(x_k)_{k \geq p}$ is equal to **infinity**, and we write $x_k \to \infty$ (as $k \to \infty$), or we write $\lim_{k \to \infty} x_k = \infty$, when

$$\forall r \in \mathbf{R}\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ (k \geq m \implies x_k > r)$$

Similarly, we say that $(x_k)_{k \geq p}$ **diverges to** $-\infty$, or that the limit of $(x_k)_{k \geq p}$ is equal to **negative infinity**, and we write $x_k \to -\infty$ (as $k \to \infty$), or we write $\lim_{k \to \infty} x_k = -\infty$ when

$$\forall r \in \mathbf{R}\ \exists m \in \mathbf{Z}\ \forall k \in \mathbf{Z}_{\geq p}\ (k \geq m \implies x_k < r)$$

REMARK 1.2.3

We shall assume that students are familiar with sequences and limits of sequences from first-year calculus. For example, students should know that if the limit of a sequence exists, then it is unique. Also, the limit does not depend on the first few terms (indeed the first few finitely many terms) and so we often omit the starting value p from our notation and write the sequence $(x_k)_{k \geq p}$ as (x_k). Students should also be able to calculate limits using various limit rules, such as Operation on Limits, the Comparison Theorem, and the Squeeze Theorem (which can all be found in the Appendix).

DEFINITION 1.2.4: Bounded above, Bounded below, Bounded

Let (x_k) be a sequence in **R**. For $b \in \mathbf{R}$, we say that the sequence (x_k) is **bounded above** by b when the set $\{x_k\}$ is bounded above by b; that is, when $x_k \leq b$ for all k, and we say that the sequence (x_k) is **bounded below** by b when the set $\{x_k\}$ is bounded below by b; that is, when $b \leq x_k$ for all k. We say (x_k) is **bounded above** when it is bounded above by some element $b \in \mathbf{R}$, we say that (x_k) is bounded below when it is bounded below by some $b \in \mathbf{R}$, and we say that (x_k) is **bounded** when it is bounded above and bounded below.

DEFINITION 1.2.5: Increasing, Non-decreasing, Strictly increasing, Strictly decreasing, Monotonic

Let $(x_k)_{k \geq p}$ be a sequence in **R**.

- (x_k) is **increasing (non-decreasing)** when

$$\forall k, \ell \in \mathbf{Z}_{\geq p} \ (k \leq \ell \implies x_k \leq x_\ell)$$

- (x_k) is **strictly increasing** when

$$\forall k, \ell \in \mathbf{Z}_{\geq p} \ (k < \ell \implies x_k < x_\ell)$$

- (x_k) is **decreasing (non-increasing)** when

$$\forall k, \ell \in \mathbf{Z}_{\geq p} \ (k \leq \ell \implies x_k \geq x_\ell)$$

- (x_k) is **strictly decreasing** when

$$\forall k, \ell \in \mathbf{Z}_{\geq p} \ (k < \ell \implies x_k > x_\ell)$$

- (x_k) is **monotonic** when it is either increasing or decreasing.

THEOREM 1.2.6: Monotonic Convergence Theorem

Let (x_k) be a sequence in **R**.

(1) *Suppose (x_k) is increasing. If (x_k) is bounded above, then $x_k \to \sup\{x_k\}$, and if (x_k) is not bounded above, then $x_k \to \infty$.*

(2) *Suppose (x_k) is decreasing. If (x_k) is bounded below, then $x_k \to \inf\{x_k\}$, and if (x_k) is not bounded below, then $x_k \to -\infty$.*

Proof of: Theorem 1.2.6

We prove (1). Let (x_k) be an increasing sequence. Assume (x_k) is bounded above, say by $b \in \mathbf{R}$. Let $A = \{x_k | k \geq p\}$ (so A is the range of the sequence (x_k)). Note that A is non-empty and bounded above (indeed b is an upper bound for A). By Theorem 1.1.8 (1), A has a supremum in **R**. Let $a = \sup\{x_k \mid k \geq p\}$. Note that $a \geq x_k$ for all $k \geq p$ and $a \leq b$ by the definition of supremum. Let $\varepsilon > 0$. By Theorem 1.1.9 (1), we can choose an index $m \geq p$ so that $x_m \in A$ satisfies $a - \varepsilon < x_m \leq a$. Since (x_k) is increasing, for all $k \geq m$, we have $x_k \geq x_m$, so we have $a - \varepsilon < x_m \leq x_k \leq a$, and hence $|x_k - a| < \varepsilon$. Thus, $\lim\limits_{k \to \infty} x_k = a \leq b$.

DEFINITION 1.2.7

For $a, b \in \mathbf{R}$ with $a \leq b$, we write

- $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$
- $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$
- $(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$
- $[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$
- $(a, \infty) = \{x \in \mathbf{R} \mid a < x\}$
- $[a, \infty) = \{x \in \mathbf{R} \mid a \leq x\}$
- $(-\infty, b) = \{x \in \mathbf{R} \mid x < b\}$
- $(-\infty, b] = \{x \in \mathbf{R} \mid x \leq b\}$
- $(-\infty, \infty) = \mathbf{R}$

An **interval** in $\mathbf{R}$ is any set of one of the above forms.

- **Degenerate** intervals: If $a = b$, then $(a, b) = [a, b) = (a, b] = \emptyset$, and $[a, b] = \{a\}$.
- **Non-degenerate** intervals contain at least two points.
- **Open** intervals: $\emptyset$, (a, b), (a, ∞), $(-\infty, b)$, and $(-\infty, \infty)$.
- **Closed** intervals: $\emptyset$, $[a, b]$, $[a, \infty)$, $(-\infty, b]$, and $(-\infty, \infty)$.
- **Bounded** intervals: $\emptyset$, (a, b), $(a, b]$, $[a, b)$, and $[a, b]$.
- **Unbounded** intervals: (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, and $(-\infty, \infty)$.

THEOREM 1.2.8: Nested Interval Theorem

Let $I_1, I_2, I_3, \ldots$ be non-empty, closed, and bounded intervals in $\mathbf{R}$.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \implies \bigcap_{k \geq 1} I_k \neq \emptyset$$

Proof of: Theorem 1.2.8

For each $k \geq 1$, let $I_k = [a_k, b_k]$ with $a_k \leq b_k$. For each k, since $I_{k+1} \subseteq I_k$, we have $a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$. Since $a_k \leq a_{k+1}$ for all k, the sequence (a_k) is increasing. Since $a_k \leq b_k \leq b_{k-1} \leq \cdots \leq b_1$ for all k, the sequence (a_k) is bounded above by b_1. Since (a_k) is increasing, and bounded above, it converges. Let $a = \sup\{a_k\} = \lim_{k \to \infty} a_k$. Similarly, (b_k) is decreasing, and bounded below by a_1, so it converges. Let $b = \inf\{b_k\} = \lim_{k \to \infty} b_k$. Since $a_k \leq b_k$ for all k, by the Comparison Theorem, we have $a \leq b$, and so the interval $[a, b]$ is not empty. Since (a_k) is increasing, with $a_k \to a$, it follows (proof as an exercise), that $a_k \leq a$ for all $k \geq 1$, Similarly, $b_k \geq b$ for all $k \geq 1$, and so $[a, b] \subseteq [a_k, b_k] = I_k$. Therefore,

$$[a, b] \subseteq \bigcap_{k \geq 1} I_k \implies \bigcap_{k \geq 1} I_k \neq \emptyset$$